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ON THE VOLUME OF SETS HAVING CONSTANT WIDTH†

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ABSTRACT
A lower bound is given for the volume of sets of constant width.

1. Introduction

A set of constant width d in Euclidean space $\mathbf{R}^n$ is a compact, convex set, such that the distance between any distinct, parallel supporting hyperplanes of it is d (see [3, pp. 122–131], [2]).

The Blaschke–Lebesgue theorem states that of all planar sets having constant width d the Reuleaux triangle has the least area, $\frac{1}{2}(\pi - \sqrt{3})d^2$. The problem of determining the minimal volume of sets having constant width d in $\mathbf{R}^n$, $n > 2$, seems considerably more difficult. Lower bounds for it have been given by Firey [4] and Chakerian [1].

Let W be a set of constant width d and circumradius r in $\mathbf{R}^n$. In this note we prove the lower bound

$$\text{Vol } W \geqq \left(\sqrt{5 - 4\frac{r^2}{d^2}} - 1 \right)^n \text{Vol } B(0, d/2), \tag{1.1}$$

which implies

$$\text{Vol } W \geqq \left(\sqrt{3 + \frac{2}{n+1}} - 1 \right)^n \text{Vol } B(0, d/2). \tag{1.2}$$

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Here Vol denotes the n-dimensional volume in $\mathbf{R}^n$ and $B(\mathbf{x}, \rho)$ is the ball having center $\mathbf{x}$ and radius ρ. This bound is, for $n > 4$, an improvement over those previously known.

We also prove

THEOREM 1. *Let K be a set of constant width d and circumradius r in $\mathbf{R}^n$ having the origin 0 as the center of its circumsphere, then $K \cup -K$ contains the ball of radius $\sqrt{5(d/2)^2 - r^2} - (d/2)$ around the origin.*

This result can be seen as a relative to the well known theorem stating that the insphere of a set of constant width d is concentric to the circumsphere and its radius is $d - r$, where r is the circumradius (see [3, p. 125]).

Arguments analogous to those below, but dealing with subsets of the unit sphere, are used in [5], where an upper bound is given for the number of directions sufficient to illuminate the boundary of sets having constant width.

2. For a set $A \subset \mathbf{R}^n$ and for $\lambda > 0$ we denote by A^λ the intersection of all the balls of radius λ, having centers in A:

$$A^\lambda = \bigcap_{\mathbf{x} \in A} B(\mathbf{x}, \lambda) = \{\mathbf{p} \in \mathbf{R}^n \mid B(\mathbf{p}, \lambda) \supset A\}.$$

We also use

$$h(A, \mathbf{x}) = \sup_{\mathbf{y} \in A} \mathbf{y} \cdot \mathbf{x} \qquad \text{(the support function of } A\text{)},$$

$$\rho(A, \mathbf{x}) = \inf\{t > 0 \mid t\mathbf{x} \notin A\}.$$

Define

$$g(\lambda, r, t) = \sqrt{\lambda^2 - r^2 + t^2} - t.$$

Notice that $g(\lambda, r, t)$ is monotonic decreasing, positive and strictly convex as a function of t when $\lambda > r$.

LEMMA 1. *Let K be a nonempty set contained in the ball of radius r around the origin in $\mathbf{R}^n$, then the relation*

$$\rho(K^\lambda, \mathbf{u}) \geqq g(\lambda, r, h(K, -\mathbf{u})) \tag{2.1}$$

is satisfied for every $\lambda \geqq r$ and every $\mathbf{u} \in S^{n-1}$.

PROOF. Let $\mathbf{u}$ be any unit vector, let λ satisfy $\lambda \geqq r$ and let a be the right hand side of (2.1). We first show that $a\mathbf{u} \in K^\lambda$. Let $\mathbf{x}$ be any point of K. We have

$$\|\mathbf{x}\| \leqq r, \qquad -\mathbf{x}\cdot\mathbf{u} \leqq h(K, -\mathbf{u}).$$

Using this and $a \geqq 0$ we obtain

$$\|\mathbf{x} - a\mathbf{u}\|^2 = \|\mathbf{x}\|^2 - 2a\mathbf{x}\cdot\mathbf{u} + a^2 \leqq r^2 + 2ah(K, -\mathbf{u}) + a^2 = \lambda^2.$$

This means that $a\mathbf{u} \in B(\mathbf{x}, \lambda)$ and since $\mathbf{x}$ is an arbitrary point of K, we have

$$a\mathbf{u} \in \bigcap_{\mathbf{x}\in K} B(\mathbf{x}, \lambda) = K^\lambda.$$

The origin is also a point of K^λ, because $K \subset B(0, r)$ and $\lambda \geqq r$. K^λ is obviously convex, so we have

$$\{t\mathbf{u} \mid 0 \leqq t \leqq a\} \subset K^\lambda.$$

This shows that $\rho(K^\lambda, \mathbf{u}) \geqq a$, as needed. ∎

In some contexts, a good way to present the volume of a set $K \subset \mathbf{R}^n$ is to specify the radius of the ball having the same volume as K. We will call it the *effective radius* of the set K and denote it by er K:

$$\operatorname{Vol} K = \operatorname{Vol} B(0, \operatorname{er} K).$$

By μ we denote the $n-1$ dimensional surface area measure on S^{n-1}, the boundary of the unit ball.

THEOREM 2. *Let K be a set of diameter d and circumradius r. Let λ satisfy $\lambda > r$, then*

$$\operatorname{er} K^\lambda \geqq g(\lambda, r, d/2).$$

PROOF. As we know, K^λ contains the origin and is convex. We can therefore rewrite its volume thus:

$$\operatorname{Vol} K^\lambda = \frac{1}{n}\int_{S^{n-1}} \rho(K^\lambda, \mathbf{u})^n d\mu(\mathbf{u}).$$

Using the lemma we have

$$\operatorname{Vol} K^\lambda \geqq \frac{1}{n}\int_{S^{n-1}} g(\lambda, r, h(K, -\mathbf{u}))^n d\mu(\mathbf{u})$$

$$= \frac{1}{n}\int_{S^{n-1}} \left(\tfrac{1}{2} g(\lambda, r, h(K, -\mathbf{u}))^n + \tfrac{1}{2} g(\lambda, r, h(K, \mathbf{u}))^n\right) d\mu(\mathbf{u}).$$

Since $g(\lambda, r, t)$ is positive and convex in t, so is $g(\lambda, r, t)^n$. Therefore the above inequality implies

$$\text{(2.2)} \qquad \operatorname{Vol} K^\lambda \geqq \frac{1}{n} \int_{S^{n-1}} g(\lambda, r, \tfrac{1}{2}h(K, -\mathbf{u}) + \tfrac{1}{2}h(K, \mathbf{u}))^n d\mu(\mathbf{u}).$$

Since K has diameter d, we have

$$\tfrac{1}{2}h(K, -\mathbf{u}) + \tfrac{1}{2}h(K, \mathbf{u}) \leqq \tfrac{1}{2}d.$$

From (2.2) and the decreasing monotonicity of $g(\lambda, r, t)$ in t, we can therefore conclude that

$$\operatorname{Vol} K^\lambda \geqq \frac{1}{n} \int_{S^{n-1}} g(\lambda, r, \tfrac{1}{2}d)^n d\mu(\mathbf{u}) = \operatorname{Vol} B(0, g(\lambda, r, d/2)). \qquad ■$$

PROOF OF (1.1), (1.2). Since $K^d = K$ for sets of constant width d (see [3, p. 123]), (1.1) can be derived easily from Theorem 2 using $\lambda = d$, $W = K$. (1.2) is a consequence of (1.1) and Jung's Theorem $r \leqq d\sqrt{n/(2n+2)}$ (see [3, p. 111]). ■

Let us denote by r_n the minimal effective radius of all sets having constant width two† in $\mathbf{R}^n$. (1.2) is equivalent to $r_n \geqq \sqrt{3 + 2/(n+1)} - 1$. From the proof it is evident that equality does not occur when $n > 1$. As mentioned above, the exact computation of r_n, for $n \geqq 3$, is probably very hard, however, the following problems seem to be answerable.

PROBLEM 1. Is the sequence $\{r_n\}$ monotonic decreasing?

PROBLEM 2. Show that $\lim_{n \to \infty} r_n$ exists and compute it.

Inequality (1.2) shows that $\liminf r_n \geqq \sqrt{3} - 1$. Because the unit ball has the largest volume among all sets having constant width 2 (see [3, pp. 106–107]), we have $\limsup r_n \leqq 1$. As far as we know any value between $\sqrt{3} - 1$ and 1 is a possible candidate for $\lim r_n$. (If the answer to Problem 1 is 'yes' then surely $\limsup r_n \leqq r_2 < 1$.)

3. We now prove a generalization of Theorem 1.

† The Blaschke selection principle implies the minimum is attained.

THEOREM 3. *Let K be a set of diameter d contained in the ball $B(0, r)$ in $\mathbf{R}^n$. Let λ satisfy $\lambda > r$, then*

$$K^\lambda \cup -K^\lambda \supset B(0, g(\lambda, r, d/2)).$$

PROOF. Let $\mathbf{u} \in S^{n-1}$. Because K has diameter d, we have $h(K, \mathbf{u}) + h(K, -\mathbf{u}) \leqq d$. Therefore

$$\min\{h(K, \mathbf{u}), h(K, -\mathbf{u})\} \leqq \tfrac{1}{2}d.$$

Using obvious properties of $\rho(\cdot, \cdot)$, Lemma 1 and the fact that $g(\lambda, r, t)$ is monotonic decreasing in t, we get

$$\begin{aligned} \rho(K^\lambda \cup -K^\lambda, \mathbf{u}) &\geqq \max\{\rho(K^\lambda, \mathbf{u}), \rho(-K^\lambda, \mathbf{u})\} \\ &= \max\{\rho(K^\lambda, \mathbf{u}), \rho(K^\lambda, -\mathbf{u})\} \\ &\geqq \max\{g(\lambda, r, h(K, -\mathbf{u})), g(\lambda, r, h(K, \mathbf{u}))\} \\ &= g(\lambda, r, \min\{h(K, \mathbf{u}), h(K, -\mathbf{u})\}) \\ &\geqq g(\lambda, r, d/2). \end{aligned}$$

This proves $K^\lambda \cup -K^\lambda \supset B(0, g(\lambda, r, d/2))$ as needed. ∎

PROOF OF THEOREM 1. Since $K^d = K$, using Theorem 3 with $\lambda = d$ gives Theorem 1.

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